

12.1 # 4. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, where $u = \sin kt \cos kx$

The partial derivatives $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$ exist, since:

$$\frac{\partial u}{\partial t} = kc \cos kt \cos kx$$

$$\bullet \frac{\partial^2 u}{\partial t^2} = -k^2 c^2 \sin kt \cos kx, \text{ exists.}$$

$$\frac{\partial u}{\partial x} = -k \sin kt \sin kx$$

$$\bullet \frac{\partial^2 u}{\partial x^2} = -k^2 \sin kt \cos kx, \text{ exists.}$$

$$\begin{aligned} \text{And } c^2 \frac{\partial^2 u}{\partial x^2} &= c^2 (-k^2 \sin kt \cos kx) \\ &= -k^2 c^2 \sin kt \cos kx \\ &= \frac{\partial^2 u}{\partial t^2} \end{aligned}$$

Then u is a solution of the PDE
for any constants k and c . ■

$$12.1 \# 15. \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\text{where } u(x, y) = a \ln(x^2 + y^2) + b$$

$$\frac{\partial^2 u}{\partial x^2} \text{ and } \frac{\partial^2 u}{\partial y^2} \text{ exist:}$$

$$\rightarrow \frac{\partial u}{\partial x} = \frac{2ax}{x^2 + y^2}$$

$$\rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{2a(x^2 + y^2) - 2x(2ax)}{(x^2 + y^2)^2}$$

$$= \frac{-2ax^2 + 2ay^2}{(x^2 + y^2)^2}$$

$$\rightarrow \frac{\partial u}{\partial y} = \frac{2ay}{x^2 + y^2}$$

$$\rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{2a(x^2 + y^2) - 2y(2ay)}{(x^2 + y^2)^2}$$

$$= \frac{-2ay^2 + 2ax^2}{(x^2 + y^2)^2}$$

$$\text{Then we have } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus u is a solution of the PDE.

$$* (u = 110, x^2 + y^2 = 1) \text{ and } (u = 0, x^2 + y^2 = 100)$$

$$\Rightarrow a \ln(1) + b = 110$$

$$\Rightarrow a \cdot 0 + b = 110$$

$$\Rightarrow b = \boxed{110}$$

$$\Rightarrow a \ln(100) + b = 0$$

$$\Rightarrow 2a \ln(10) + 110 = 0$$

$$\Rightarrow a = \frac{-110}{2 \ln(10)} = \boxed{-\frac{55}{\ln(10)}}$$

$$\text{The solution becomes } u(x, y) = -\frac{55}{\ln(10)} \ln(x^2 + y^2) + 110$$

12.3 # 11.

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x$$

where $B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

$$\lambda_n = \frac{cn\pi}{L}$$

• But since the initial velocity $g = 0$, then $B_n^* = 0$

• $L = 1$

• $c^2 = 1 \Rightarrow$ take $c = 1$ (we will get the same answer for $c = -1$ since $\cos(-\frac{cn\pi}{L}) = \cos \frac{cn\pi}{L}$.)

Thus $u(x, t) = \sum_{n=1}^{\infty} B_n \cos n\pi t \sin n\pi x$

Calculate B_n , with $f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{4} \\ x - \frac{1}{4} & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2} \\ -x + \frac{3}{4} & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ 0 & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}$

$$B_n = 2 \int_0^1 f(x) \sin n\pi x dx$$

$$B_n = 2 \int_{\frac{1}{4}}^{\frac{1}{2}} (x - \frac{1}{4}) \sin(n\pi x) dx + 2 \int_{\frac{1}{2}}^{\frac{3}{4}} (-x + \frac{3}{4}) \sin(n\pi x) dx$$

$x - \frac{1}{4}$	\oplus	$\sin(n\pi x)$	$-x + \frac{3}{4}$	\oplus	$\sin(n\pi x)$
1		$-\frac{1}{n\pi} \cos(n\pi x)$	-1		$-\frac{1}{n\pi} \cos(n\pi x)$
0		$-\frac{1}{n^2\pi^2} \sin(n\pi x)$	0		$-\frac{1}{n^2\pi^2} \sin(n\pi x)$

$$B_n = 2 \left[-\left(x - \frac{1}{4}\right) \frac{1}{n\pi} \cos(n\pi x) + \frac{1}{n^2\pi^2} \sin(n\pi x) \right]_{\frac{1}{4}}^{\frac{1}{2}} + 2 \left[-\left(-x + \frac{3}{4}\right) \frac{1}{n\pi} \cos(n\pi x) - \frac{1}{n^2\pi^2} \sin(n\pi x) \right]_{\frac{1}{2}}^{\frac{3}{4}}$$

$$B_n = 2 \left[-\frac{1}{4n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2} \sin\left(\frac{n\pi}{4}\right) \right] + 2 \left[-\frac{1}{n^2\pi^2} \sin\left(\frac{3n\pi}{4}\right) + \frac{1}{4n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \right]$$

$$B_n = \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) - \frac{2}{n^2\pi^2} \sin\left(\frac{n\pi}{4}\right) - \frac{2}{n^2\pi^2} \sin\left(\frac{3n\pi}{4}\right)$$

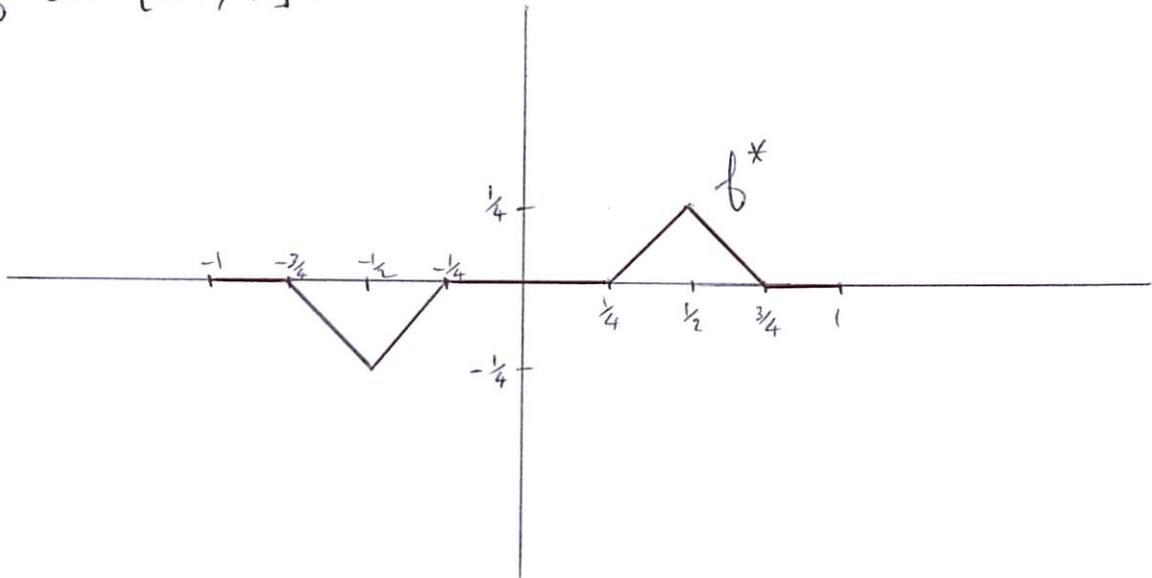
graph. \rightarrow

12.3. # 11. graph of $u(x, \frac{1}{2})$

$$u(x, t) = \frac{1}{2} \left[f^*(x - ct) + f^*(x + ct) \right] \quad \begin{cases} c=1 \\ t=\frac{1}{2} \end{cases}$$

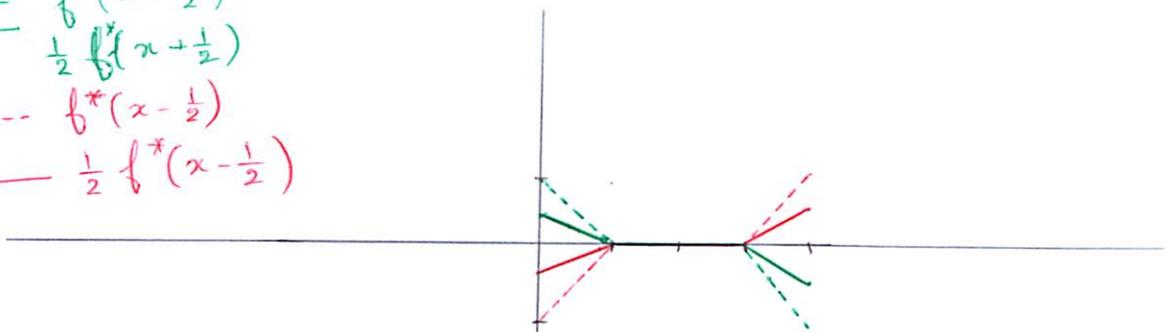
$$\text{then } u(x, \frac{1}{2}) = \frac{1}{2} \left[f^*(x - \frac{1}{2}) + f^*(x + \frac{1}{2}) \right]$$

where f^* is the periodic odd extension of f on $[-1, 1]$.

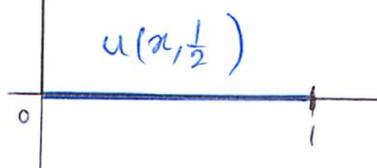


$f^*(x - \frac{1}{2})$ is the translation of f^* by $\frac{1}{2}$ to the right,
 $f^*(x + \frac{1}{2})$ is the translation of f^* by $\frac{1}{2}$ to the left,
 both considered on $[0, 1]$. Therefore:

- $f^*(x + \frac{1}{2})$
- $\frac{1}{2} f^*(x + \frac{1}{2})$
- $f^*(x - \frac{1}{2})$
- $\frac{1}{2} f^*(x - \frac{1}{2})$



Thus $u(x, \frac{1}{2}) = 0$.



12.4 # 11

$$u_{xx} + 2u_{xy} + u_{yy} = 0 \quad (*)$$

$$\Delta = AC - B^2 \quad \text{where } A=1, B=1, C=1$$

$$\Delta = 1 - 1 = 0 \quad \Rightarrow \text{parabolic PDE}$$

$$(CF): \quad Ay'^2 - 2By' + C = 0 \quad (\text{where } y' = \frac{dy}{dx})$$

$$y'^2 - 2y' + 1 = 0$$

$$(y' - 1)^2 = 0$$

$$y' = 1$$

$$\text{Then } y = x + c$$

$$D(x, y) = E(x, y) = c = y - x$$

then the new variables are

$$v = x$$

$$w = E(x, y) = y - x$$

$$\text{then } v_x = 1, v_y = 0$$

$$w_x = -1, w_y = 1$$

By chain rule:

$$\bullet \quad u_x = u_v v_x + u_w w_x = u_v - u_w$$

$$\bullet \quad u_{xx} = (u_v - u_w)_x = u_{vv} v_x + u_{vw} w_x - (u_{wv} v_x + u_{ww} w_x) \\ = u_{vv} - u_{vw} - u_{wv} + u_{ww}$$

Assuming that u and its derivatives are continuous

$$\circ \quad u_{vw} = u_{wv} \text{ then:}$$

$$u_{xx} = u_{vv} - 2u_{vw} + u_{ww}$$

$$\bullet \quad u_{xy} = (u_v - u_w)_y = u_{vv} v_y + u_{vw} w_y - (u_{wv} v_y + u_{ww} w_y)$$

$$u_{xy} = u_{vw} - u_{ww}$$

$$\bullet \quad u_y = u_v v_y + u_w w_y = u_w$$

$$\bullet \quad u_{yy} = u_{wv} v_y + u_{ww} w_y = u_{ww}$$

Substituting into (*)

$$u_{vv} - 2u_{vw} + u_{ww} + 2u_{vw} - 2u_{ww} + u_{ww} = 0$$

$\Rightarrow u_{vv} = 0$: Normal form

Integrating w.r.t. v

$$u_v = f(w)$$

Integrating w.r.t. v

$$u = v f(w) + g(w), \quad \text{where } f \text{ and } g \text{ are arbitrary } C^2 \text{-functions}$$

Therefore $u(x, y) = x f(y-x) + g(y-x) \quad \blacksquare$

$$12.4. \# 13 \quad u_{xx} + 5u_{xy} + 4u_{yy} = 0 \quad (*)$$

$$\Delta = AC - B^2, \quad A=1, \quad B=\frac{5}{2}, \quad C=4$$

$$\Delta = 4 - \left(\frac{5}{2}\right)^2 = 4 - \frac{25}{4} = \frac{16-25}{4} = \frac{-9}{4} < 0$$

So its a Hyperbolic PDE

$$(CE): \quad Ay'^2 - 2By' + C = 0 \quad \text{where } y' = \frac{dy}{dx}$$

$$y'^2 - 5y' + 4 = 0$$

$$(y'-1)(y'-4) = 0$$

$$y' = 1 \quad \text{OR} \quad y' = 4$$

$$y = x + c_1 \quad \text{OR} \quad y = 4x + c_2$$

$$c_1 = y - x \quad \& \quad c_2 = y - 4x$$

$$D(x,y) = c_1, \quad E(x,y) = c_2$$

Since its a Hyperbolic PDE, the new variables are

then $v = D$ and $w = E$

$$\boxed{v = y - x}, \quad \boxed{w = y - 4x}$$

$$v_x = -1, \quad w_x = -4$$

$$v_y = 1, \quad w_y = 1$$

By the chain rule:

$$u_x = u_v v_x + u_w w_x$$

$$= -u_v - 4u_w$$

$$u_{xx} = (-u_v - 4u_w)_x = -(u_v)_x - 4(u_w)_x$$

$$= -(u_{vv} v_x + u_{vw} w_x) - 4(u_{wv} v_x + u_{ww} w_x)$$

$$u_{xx} = u_{vv} + 4u_{vw} + 4u_{ww} + 16u_{www}$$

Assuming u and its derivatives are continuous
then $u_{vw} = u_{wv}$ and so,

$$u_{xx} = u_{vv} + 8u_{vw} + 16u_{www}$$

$$\begin{aligned} u_{xy} &= (-u_v - 4u_w)_y = -(u_v)_y - 4(u_w)_y \\ &= -(u_{vv}v_y + u_{vw}w_y) - 4(u_{wv}v_y + u_{ww}w_y) \\ &= -u_{vv} - u_{vw} - 4u_{wv} - 4u_{ww} \end{aligned}$$

Also by the assumption,

$$u_{xy} = -u_{vv} - 5u_{vw} - 4u_{ww}$$

$$u_y = u_v v_y + u_w w_y = u_v + u_w$$

Substituting in (*)

$$\begin{aligned} \cancel{u_{vv}} + 8u_{vw} + 16\cancel{u_{www}} - 5\cancel{u_{vv}} - 25u_{vw} - 20\cancel{u_{www}} \\ + 4\cancel{u_{vv}} + 8u_{vw} + 4\cancel{u_{www}} = 0 \end{aligned}$$

$$\Leftrightarrow -u_{vw} = 0 \quad \Leftrightarrow \boxed{u_{vw} = 0}$$

integrate with respect to w : $u_v = f(v)$

" " " " v : $u = \int f(v) dv + g(w)$

$$u = f(v) + g(w) \quad \text{where } f(v) = \int f(v) dv$$

$$\text{then } u(x, y) = f(y-x) + g(y-4x)$$

where f and g are arbitrary C^2 functions.